The Smaller (SALI) and the Generalized (GALI) Alignment Index Methods of Chaos Detection: Theory and Applications

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Outline

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Autonomous Hamiltonian systems

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\underbrace{q_1,q_2,\ldots,q_N}_{\text{positions}},\ \underbrace{p_1,p_2,\ldots,p_N}_{\text{momenta}})$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0)=(q_1(0),\ q_2(0),\ldots,q_N(0),\ p_1(0),\ p_2(0),\ldots,p_N(0))$$

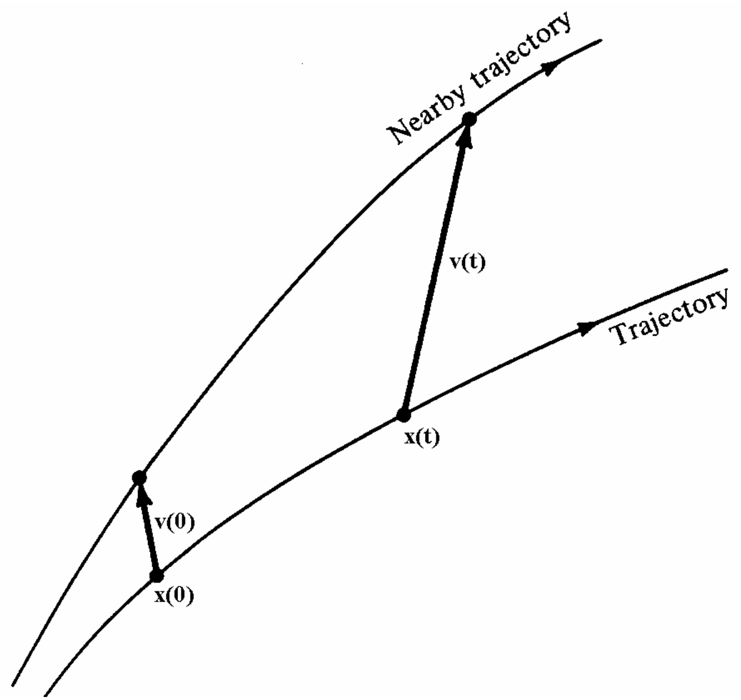
is governed by the Hamilton's equations of motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\ ,\quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Variational Equations

We use the notation $\mathbf{x} = (q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N)^T$. The deviation vector from a given orbit is denoted by

$$\mathbf{v} = (dx_1, dx_2,\ldots, dx_n)^T \text{ , with } n=2N$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of v is given by the so-called variational equations:

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}, \quad P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i,j = 1,2,\ldots,n$$

Symplectic Maps

Consider an n-dimensional symplectic map T. In this case we have discrete time.

The evolution of an orbit with initial condition

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

is governed by the equations of map T

$$P(i+1)=T\,P(i)\ ,\ i=0,1,2,\ldots$$

The evolution of an initial deviation vector

$$v(0) = (dx_1(0), dx_2(0),\ldots, dx_n(0))$$

is given by the corresponding tangent map

$$v(i+1) = \left.\frac{\partial T}{\partial P}\right|_i \cdot v(i)\ \ , i = 0, 1, 2, \ldots$$

Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\sigma(x(0), v(0)) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

Maximal Lyapunov Exponent

$\sigma_1=0 \rightarrow$ Regular motion

$\sigma_1 \neq 0 \rightarrow$ Chaotic motion

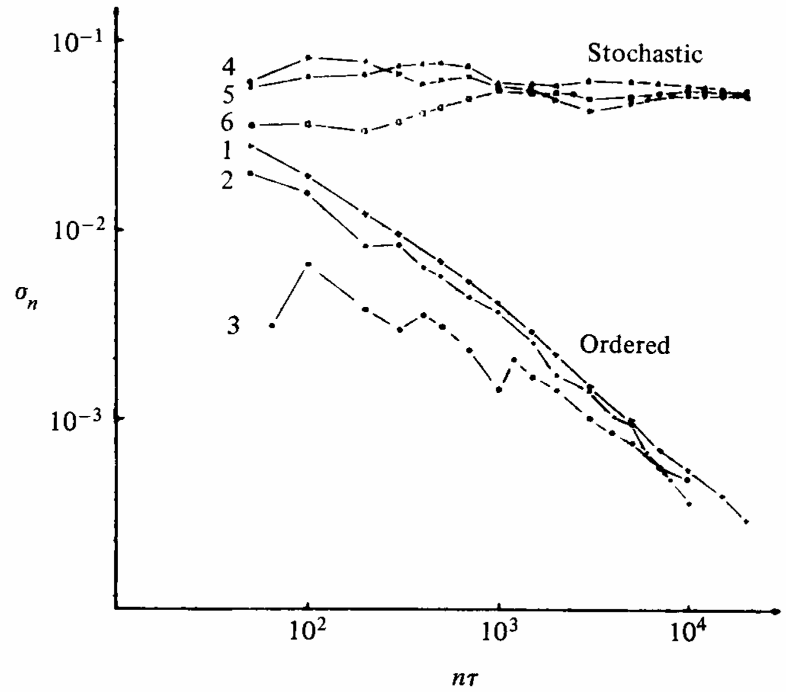


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

If we start with more than one linearly independent deviation vectors they will align to the direction defined by the largest Lyapunov exponent.

Definition of Smaller Alignment Index (SALI)

Consider the n-dimensional phase space of a conservative dynamical system (symplectic map or Hamiltonian flow).

An orbit in that space with initial condition :

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

and a deviation vector

$$v(0)=(dx_1(0), dx_2(0),\ldots, dx_n(0))$$

The evolution in time (in maps the time is discrete and is equal to the number N of the iterations) of a deviation vector is defined by:

- **the variational equations (for Hamiltonian flows) and**
- **the equations of the tangent map (for mappings)**

Definition of SALI

We follow the evolution in time of <u>two different initial deviation vectors</u> ($v_1(0)$, $v_2(0)$), and define SALI (Skokos, 2001, J. Phys. A, 34, 10029) as:

$$\mathrm{SALI}(t) = \min\left\{\left\|\hat{v}_1(t) + \hat{v}_2(t)\right\|, \left\|\hat{v}_1(t) - \hat{v}_2(t)\right\|\right\}$$

where

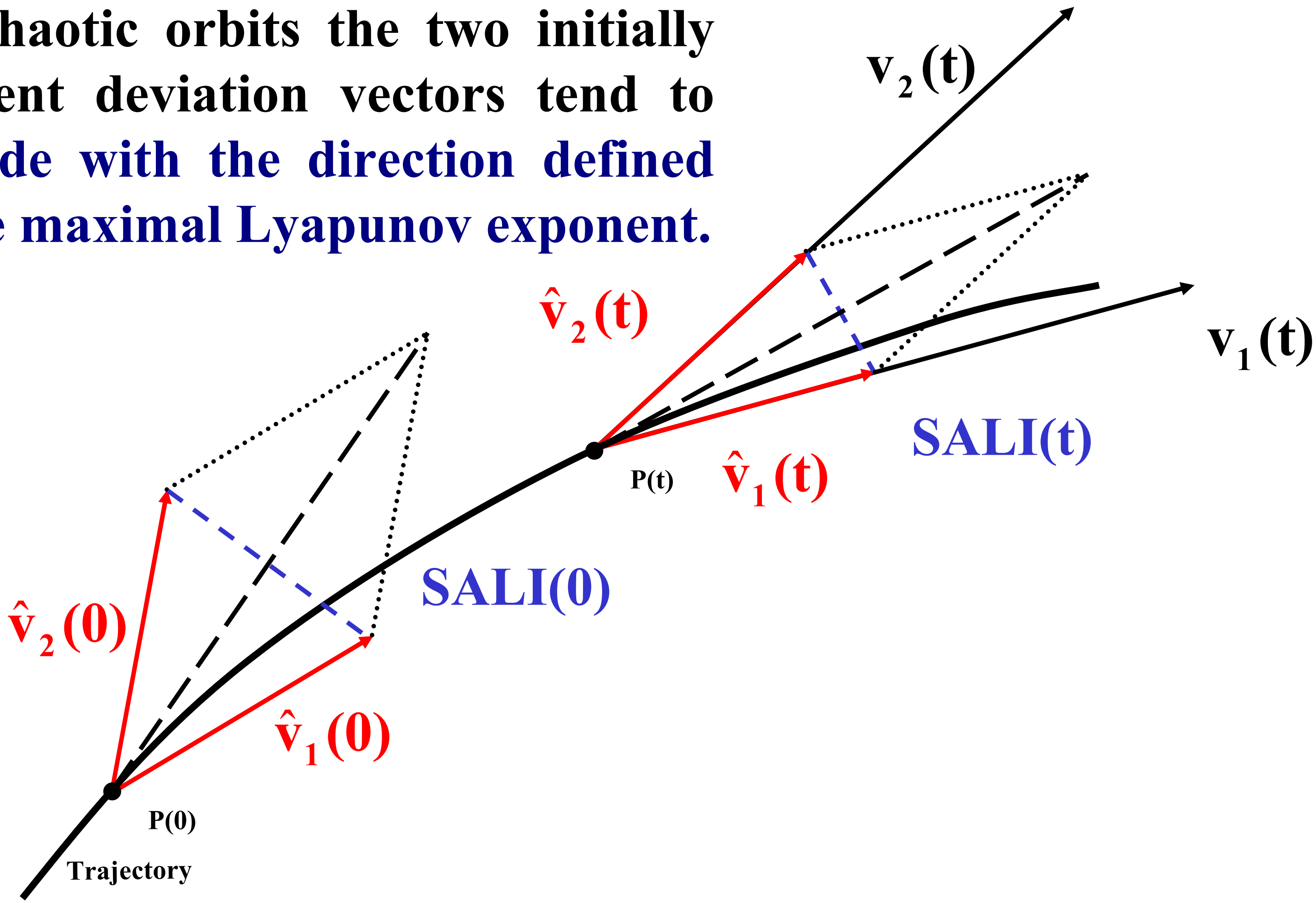
$$\hat{v}_1(t) = \frac{v_1(t)}{\left\|v_1(t)\right\|}$$

When the two vectors become collinear

$$\mathrm{SALI}(t) \rightarrow 0$$

Behavior of SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to **coincide with the direction defined by the maximal Lyapunov exponent.**

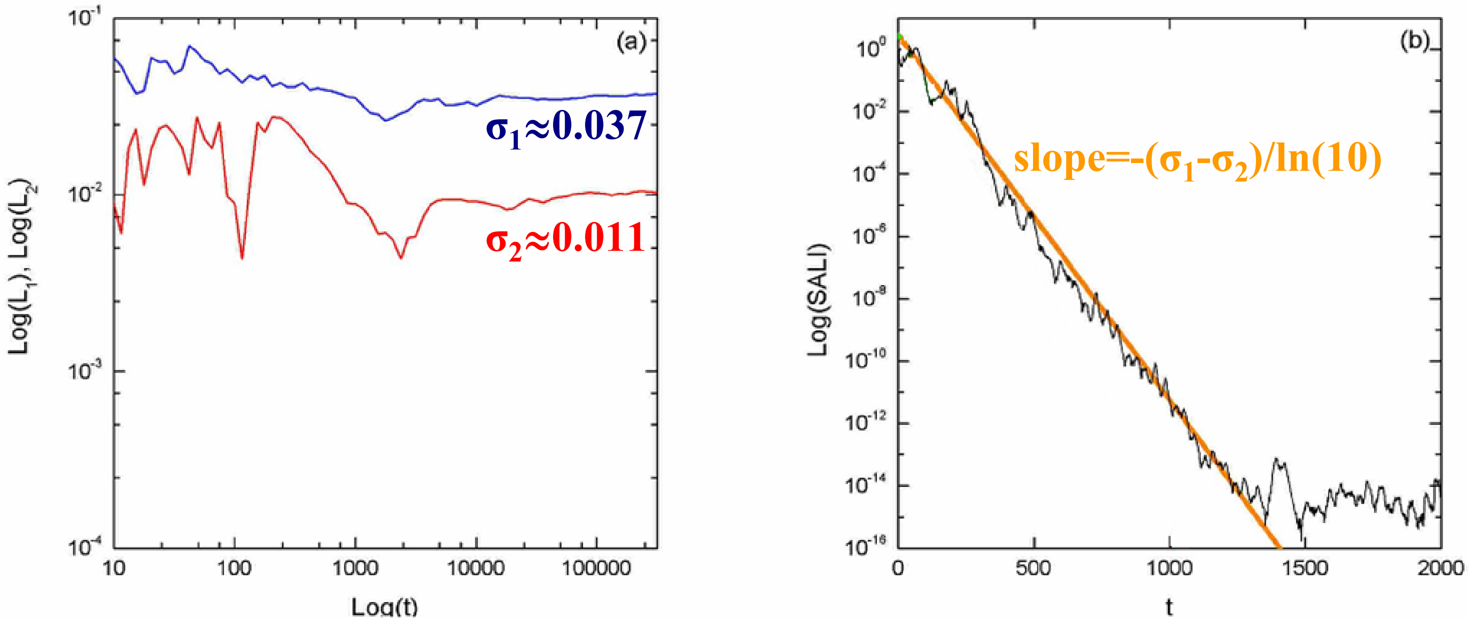


Behavior of SALI for chaotic motion

We test the validity of the approximation $SALI \propto e^{-(\sigma 1-\sigma 2)t}$ (Skokos et al., 2004, J. Phys. A, 37, 6269) for a chaotic orbit of the 3D Hamiltonian

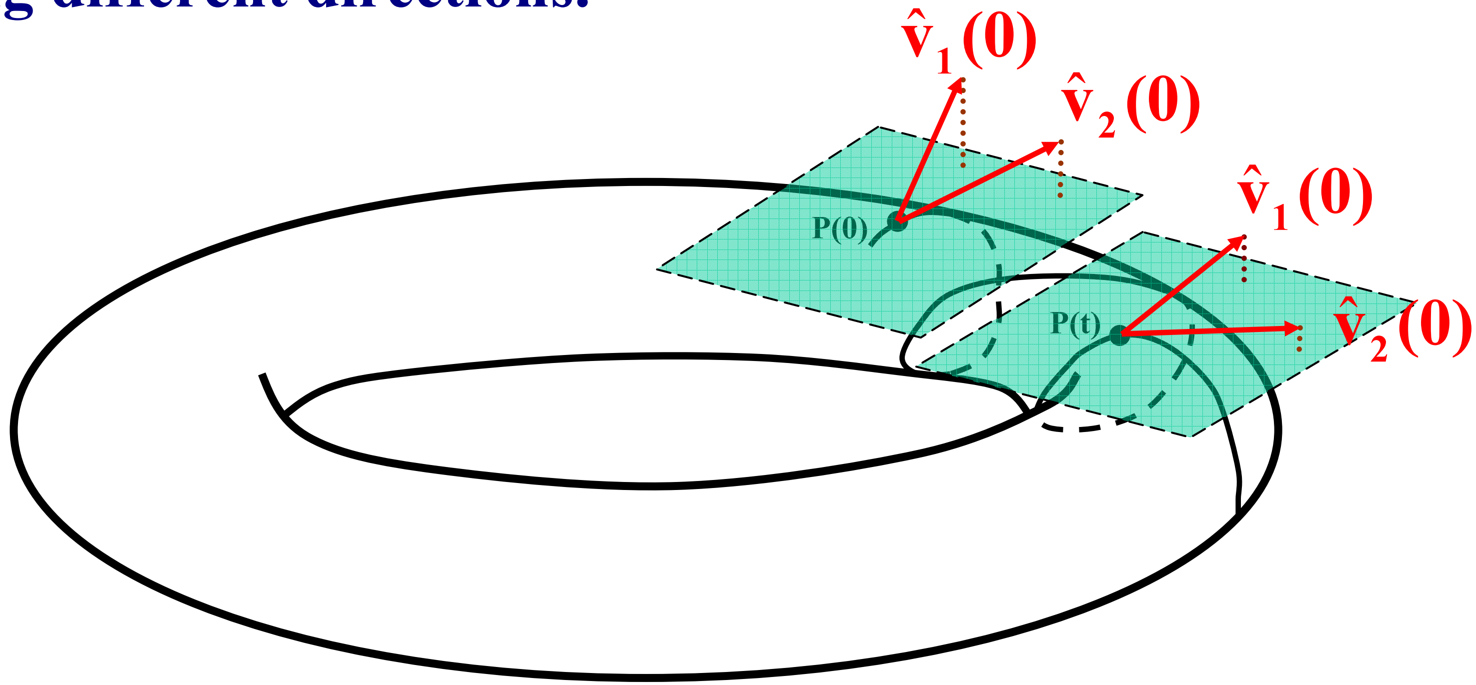
$$H = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with ω_1=1, ω_2=1.4142, ω_3=1.7321, H=0.09



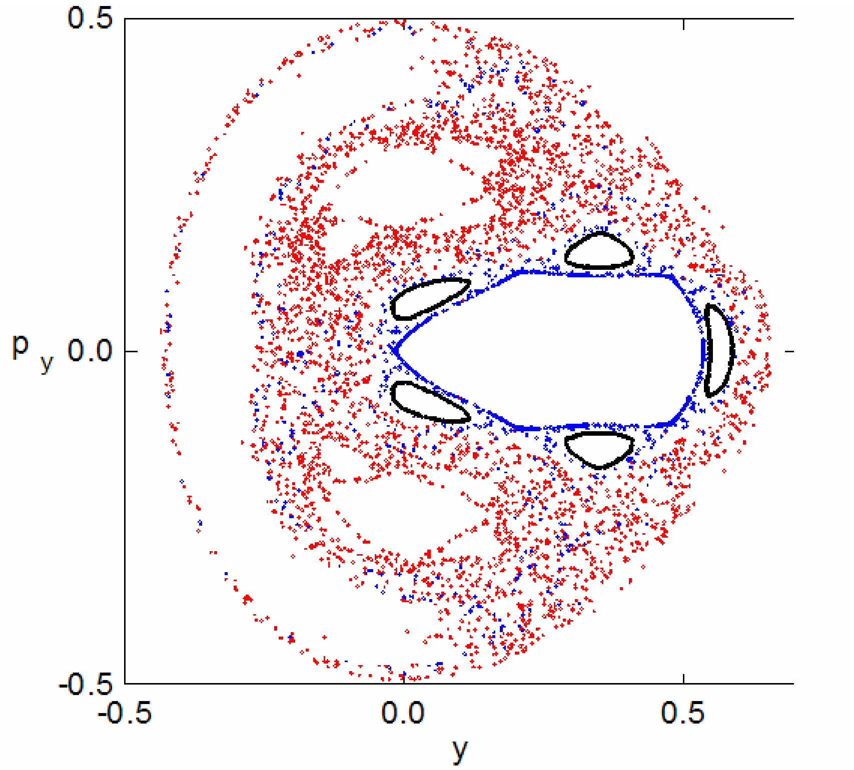
Behavior of SALI for regular motion

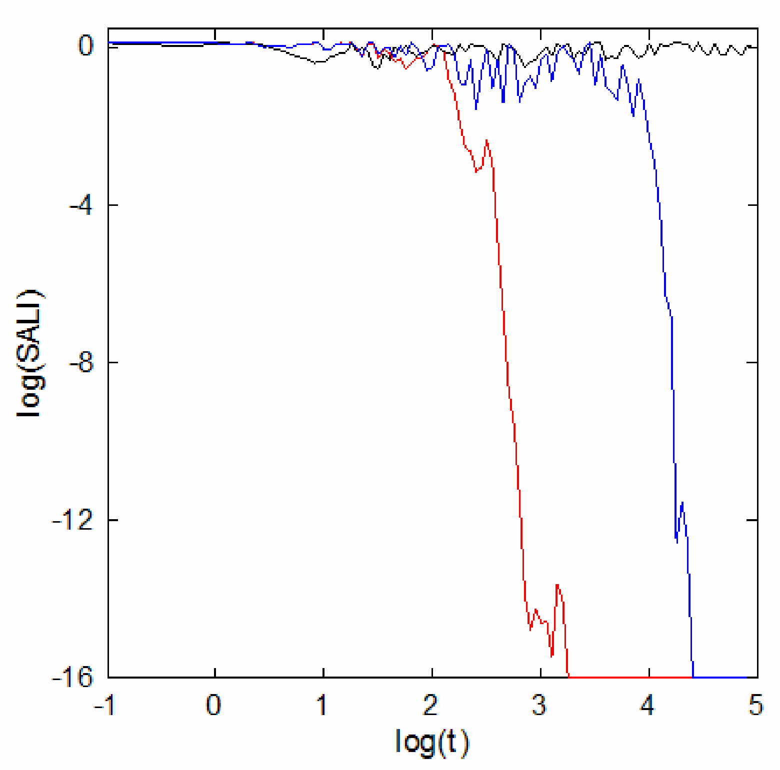
Regular motion occurs on a torus and two different initial deviation vectors become tangent to the torus, generally having different directions.



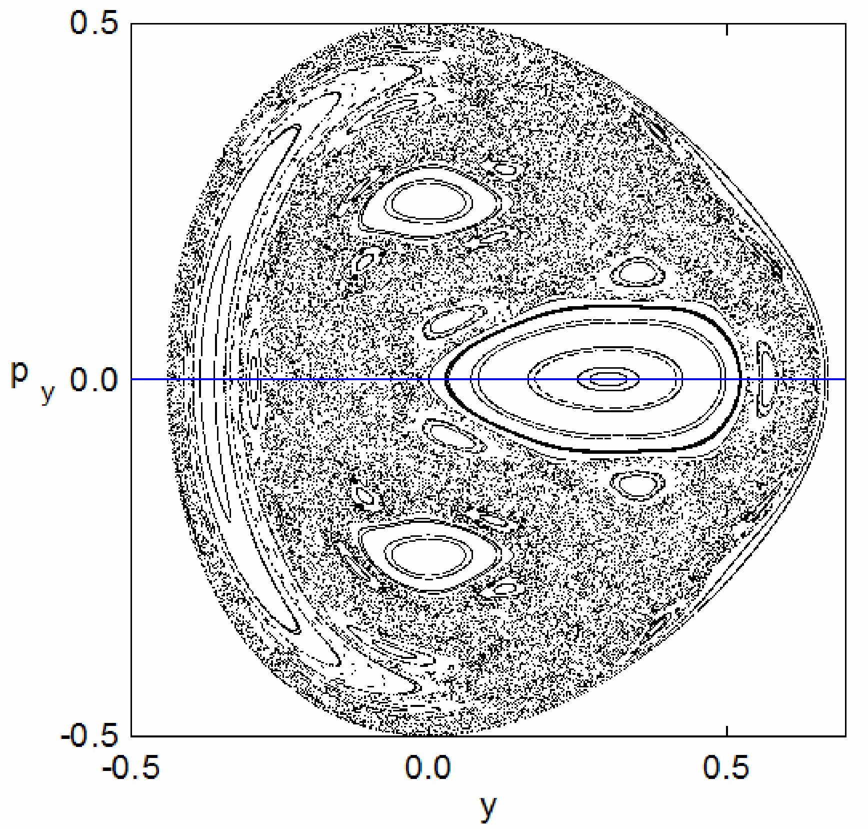
Applications – Hénon-Heiles system

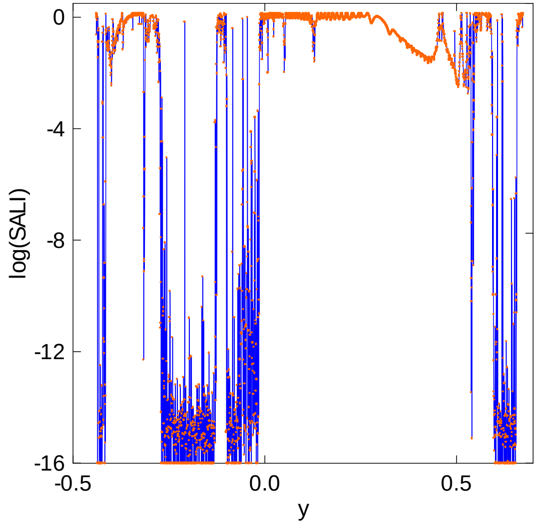
For E=1/8 we consider the orbits with initial conditions:
Ordered orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$
Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$
Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$



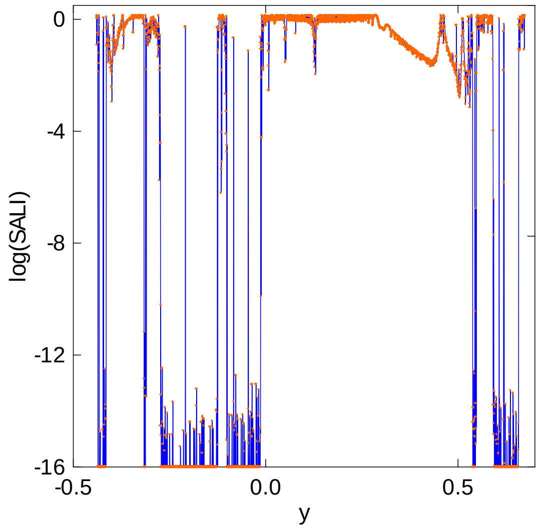


Applications – Hénon-Heiles system



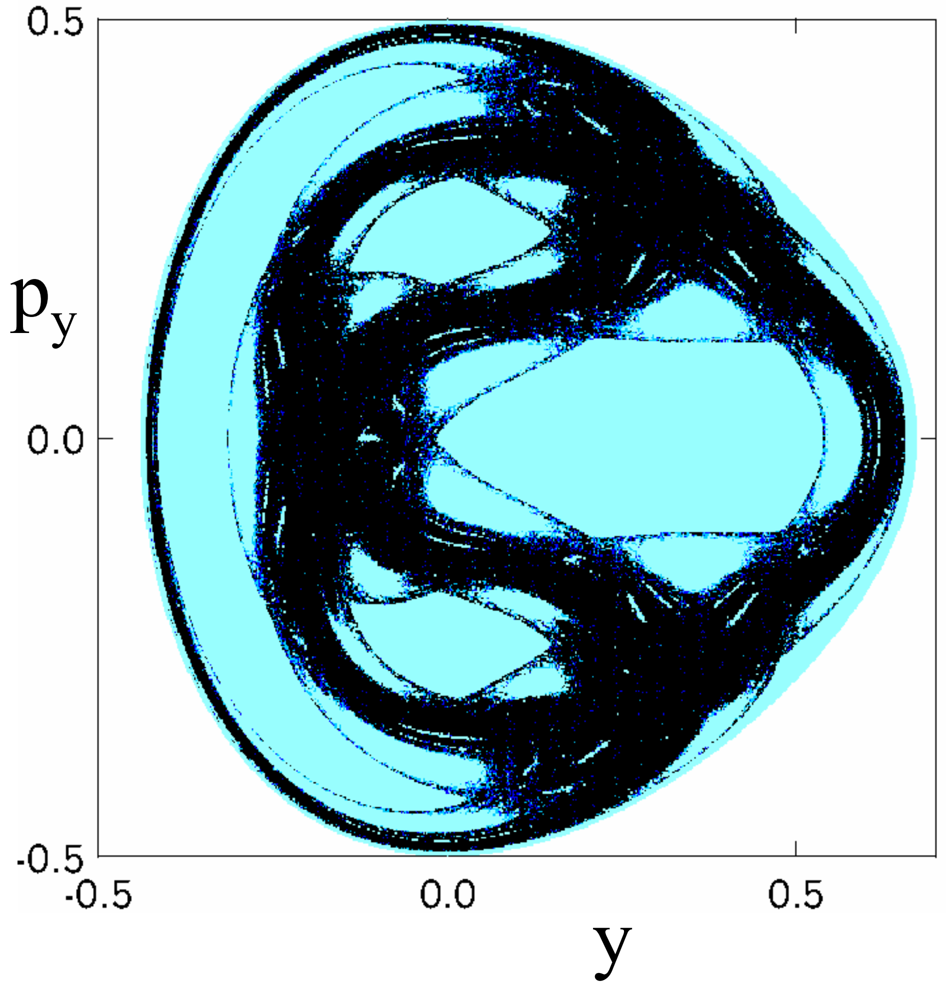


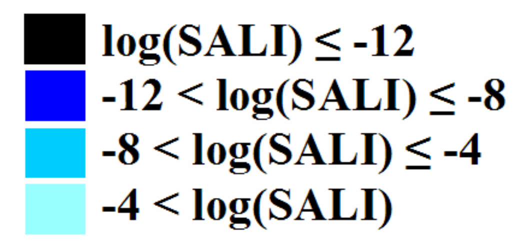
t=1000



t=4000

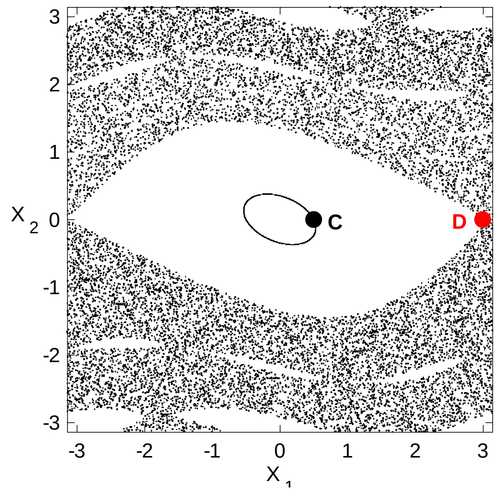
Applications – Hénon-Heiles system





Applications – 4D map

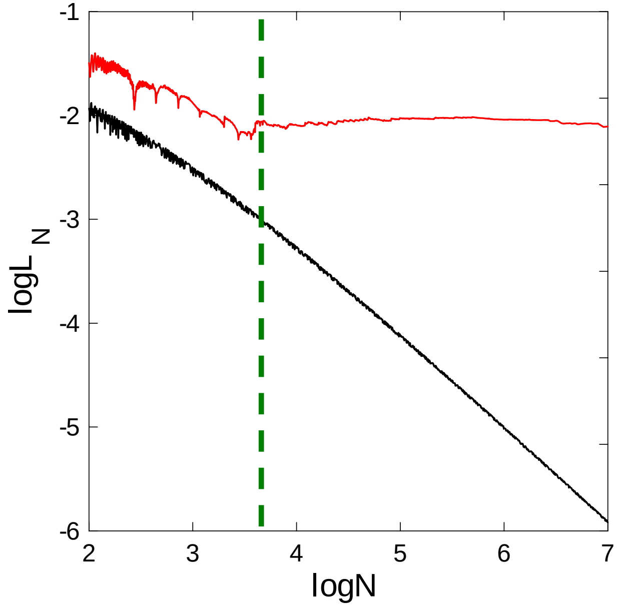
$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
x_3' &= x_3 + x_4 \\
x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)]
\end{aligned}
\quad (\text{mod } 2\pi)
$$

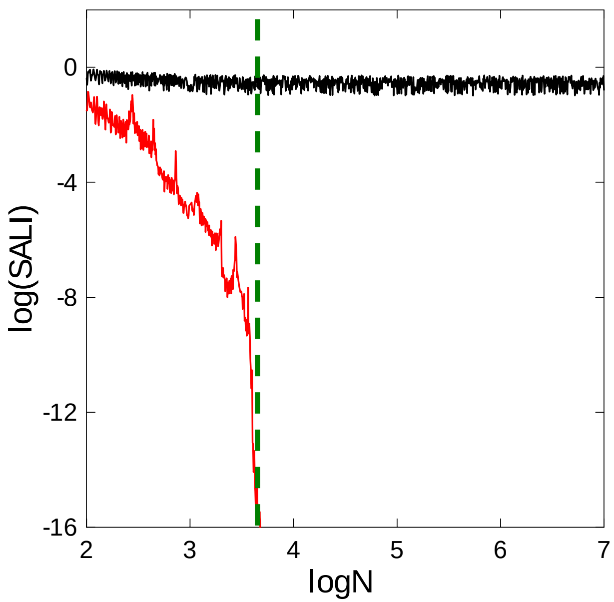


For ν=0.5, κ=0.1, μ=0.1 we consider the orbits:

ordered orbit C *with initial conditions* $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D *with initial conditions* $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.





Applications – 4D Accelerator map

We consider the 4D symplectic map

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}$$

describing the instantaneous sextupole 'kicks' experienced by a particle as it passes through an accelerator (Turchetti & Scandale 1991, Bountis & Tompaidis 1991, Vrahatis et al. 1996, 1997).

x_1 and x_3 are the particle's deflections from the ideal circular orbit, in the horizontal and vertical directions respectively.

x_2 and x_4 are the associated momenta

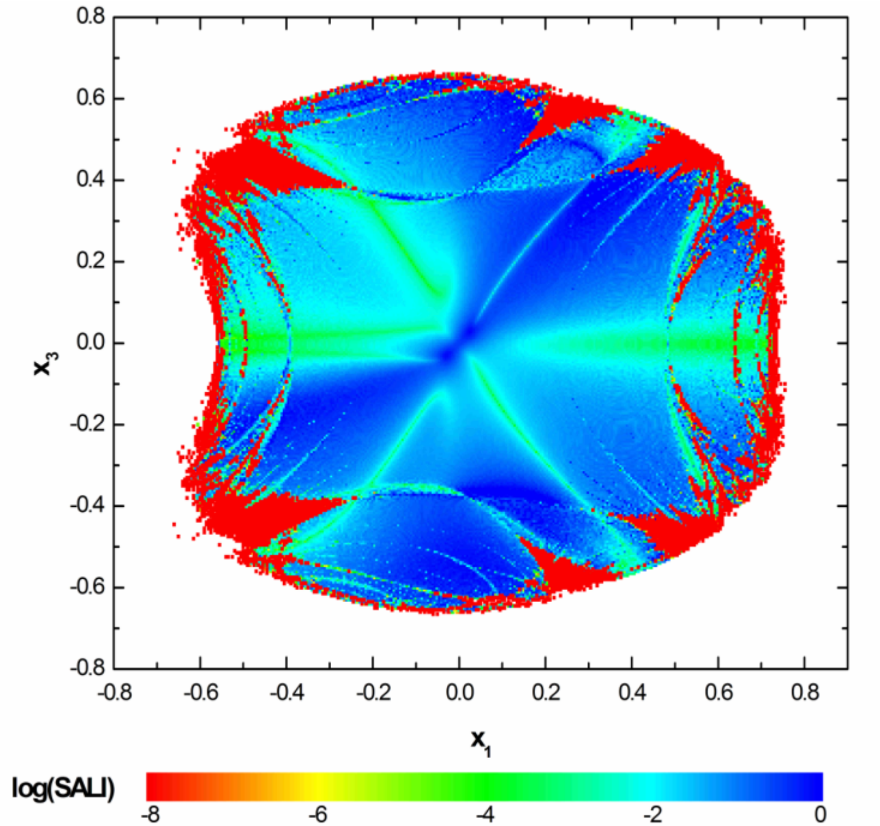
ω_1, ω_2 are related to the accelerator's tunes q_x, q_y by

$$\omega_1 = 2\pi q_x, \quad \omega_2 = 2\pi q_y$$

Our problem is to estimate the region of stability of the particle's motion, the so-called dynamic aperture of the beam (Bountis & Skokos, 2006, Nucl. Inst Meth. Phys Res. A, 561, 173).

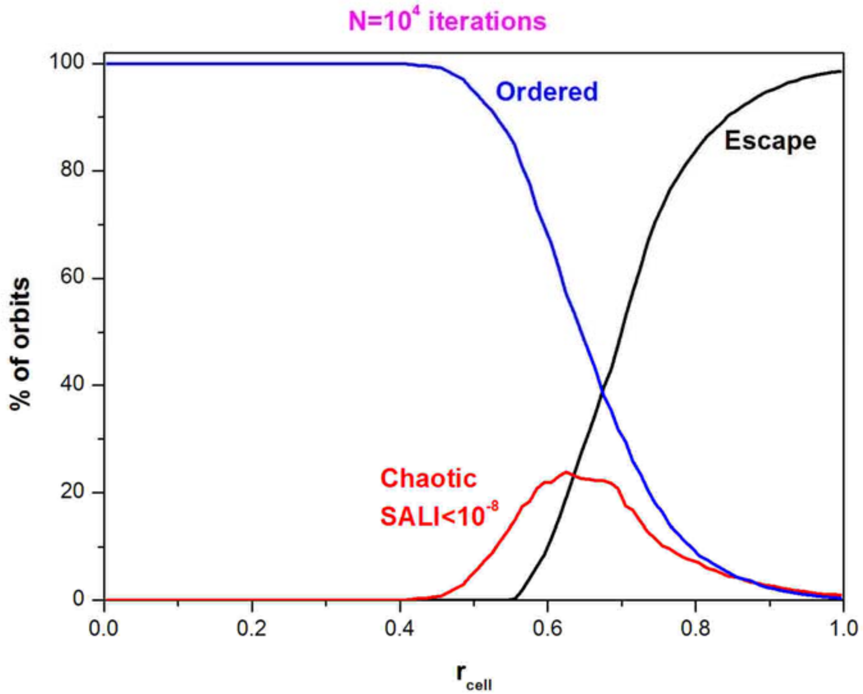
4D Accelerator map – "Global" study

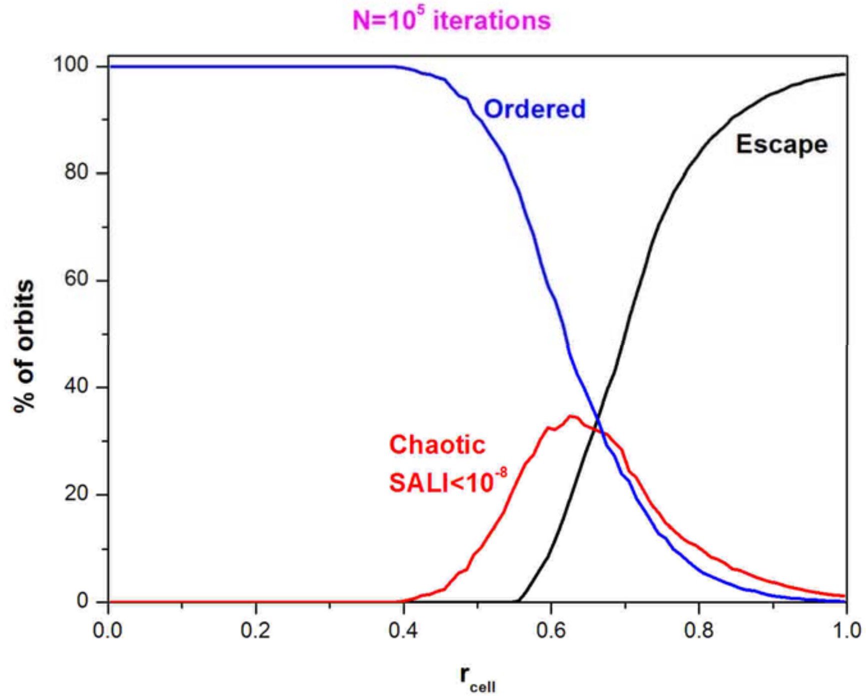
Regions of different values of the SALI on the subspace $x_2(0)=x_4(0)=0$, after 10^4 iterations (q_x=0.61803 q_y=0.4152)



4D Accelerator map – "Global" study

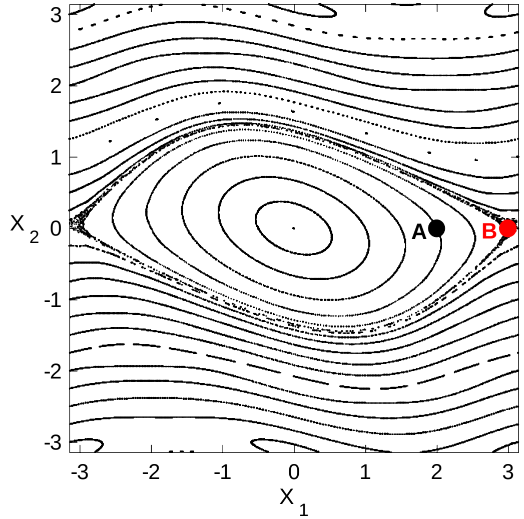
We consider 1,922,833 orbits by varying all x_1, x_2, x_3, x_4 within spherical shells of width 0.01 in a hypersphere of radius 1. (q_x=0.61803 q_y=0.4152)



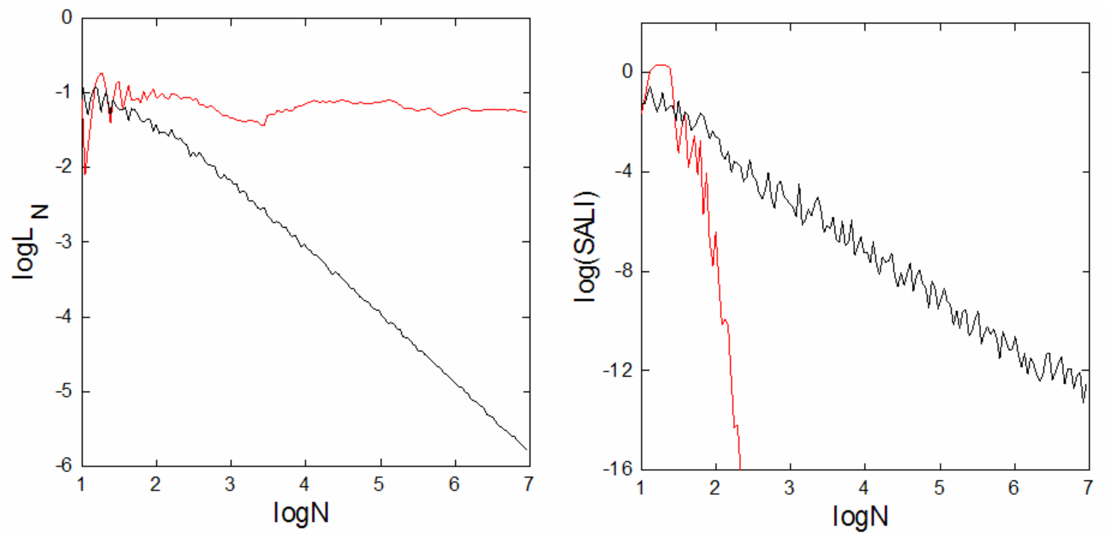


Applications – 2D map

$$x_1' = x_1 + x_2$$
$$x_2' = x_2 - \nu \sin(x_1 + x_2) \quad (\text{mod } 2\pi)$$



For ν=0.5 we consider the orbits:
ordered orbit A *with initial conditions $x_1=2$, $x_2=0$.*
chaotic orbit B *with initial conditions $x_1=3$, $x_2=0$.*



Behavior of SALI

2D maps

SALI→0 both for regular and chaotic orbits

following, however, completely different time rates which allows us to distinguish between the two cases.

Hamiltonian flows and multidimensional maps

SALI→0 for chaotic orbits

SALI→constant ≠ 0 for regular orbits

Questions

Can we generalize SALI so that the new index:

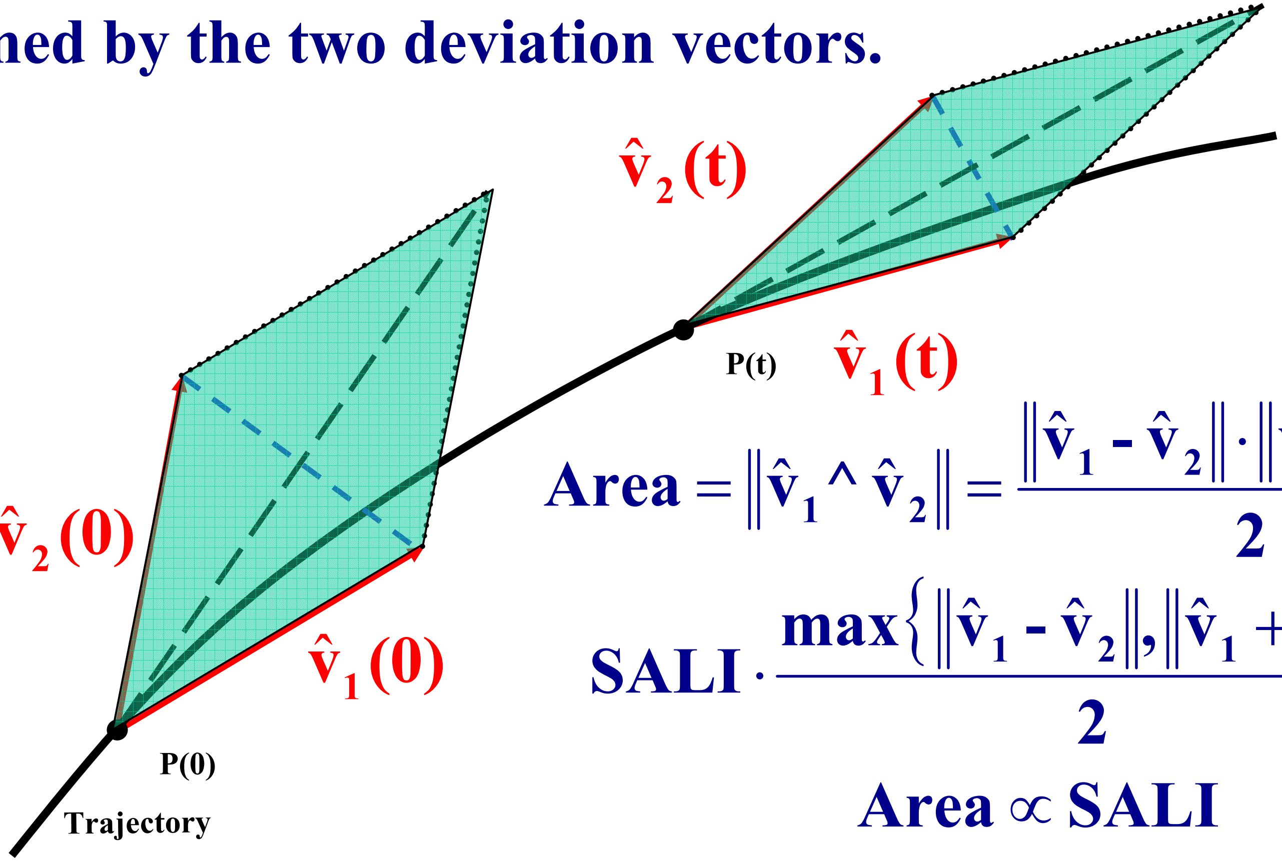
- **Can rapidly reveal the nature of chaotic orbits with $\sigma_1 \approx \sigma_2$ (SALI$\propto e^{-(\sigma1-\sigma2)t}$)?**

- **Depends on several Lyapunov exponents for chaotic orbits?**

- **Exhibits power-law decay for regular orbits depending on the dimensionality of the tangent space of the reference orbit as for 2D maps?**

Definition of Generalized Alignment Index (GALI)

SALI effectively measures the ‘area’ of the parallelogram formed by the two deviation vectors.



$$\text{Area} = \left\| \hat{v}_1 \wedge \hat{v}_2 \right\| = \frac{\left\| \hat{v}_1 - \hat{v}_2 \right\| \cdot \left\| \hat{v}_1 + \hat{v}_2 \right\|}{2} =$$

$$\text{SALI} \cdot \frac{\max\left\{ \left\| \hat{v}_1 - \hat{v}_2 \right\|, \left\| \hat{v}_1 + \hat{v}_2 \right\| \right\}}{2} \Rightarrow$$

$$\text{Area} \propto \text{SALI}$$

Definition of GALI

In the case of an N degree of freedom Hamiltonian system or a 2N symplectic map we follow the evolution of

k deviation vectors with 2≤k≤2N,

and define (Skokos et al., 2007, Physica D, 231, 30) the Generalized Alignment Index (GALI) of order k :

$$\mathrm{GALI}_k(t) = \left\| \hat{v}_1(t) \wedge \hat{v}_2(t) \wedge \ldots \wedge \hat{v}_k(t) \right\|$$

where

$$\hat{v}_1(t) = \frac{v_1(t)}{\left\| v_1(t) \right\|}$$

Wedge product

We consider as a basis of the 2N-dimensional tangent space of the Hamiltonian flow the usual set of orthonormal vectors:

$$\hat{e}_1 = (1,0,0,...,0),\ \hat{e}_2 = (0,1,0,...,0),\ ...,\ \hat{e}_{2N} = (0,0,0,...,1)$$

Then for k deviation vectors we have:

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_k \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1\,2N} \\ v_{21} & v_{22} & \cdots & v_{2\,2N} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{k\,2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{2N} \end{bmatrix}$$

$$\hat{v}_1 \wedge \hat{v}_2 \wedge \cdots \wedge \hat{v}_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2N} \begin{vmatrix} v_{1\,i_1} & v_{1\,i_2} & \cdots & v_{1\,i_k} \\ v_{2\,i_1} & v_{2\,i_2} & \cdots & v_{2\,i_k} \\ \vdots & \vdots & & \vdots \\ v_{k\,i_1} & v_{k\,i_2} & \cdots & v_{k\,i_k} \end{vmatrix} \hat{e}_{i_1} \wedge \hat{e}_{i_2} \wedge \cdots \wedge \hat{e}_{i_k}$$

Computation of GALI

For k deviation vectors:

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_k \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1\,2N} \\ v_{21} & v_{22} & \cdots & v_{2\,2N} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{k\,2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{2N} \end{bmatrix} = A \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{2N} \end{bmatrix}$$

the 'norm' of the wedge product is given by:

$$\left\| \hat{v}_1 \wedge \hat{v}_2 \wedge \cdots \wedge \hat{v}_k \right\| = \left\{ \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2N} \begin{vmatrix} v_{1\,i_1} & v_{1\,i_2} & \cdots & v_{1\,i_k} \\ v_{2\,i_1} & v_{2\,i_2} & \cdots & v_{2\,i_k} \\ \vdots & \vdots & & \vdots \\ v_{k\,i_1} & v_{k\,i_2} & \cdots & v_{k\,i_k} \end{vmatrix}^2 \right\}^{1/2} = \sqrt{\det(A \cdot A^T)}$$

Computation of GALI

From Singular Value Decomposition (SVD) of A^T we get:

$$A^T = U \cdot W \cdot V^T$$

where U is a column-orthogonal 2N×k matrix ($U^T \cdot U = I$), V^T is a k×k orthogonal matrix ($V \cdot V^T = I$), and W is a diagonal k×k matrix with positive or zero elements, the so-called singular values. So, we get:

$$\det(A \cdot A^T) = \det(V \cdot W^T \cdot U^T \cdot U \cdot W \cdot V^T) = \det(V \cdot W \cdot I \cdot W \cdot V^T) =$$

$$\det(V \cdot W^2 \cdot V^T) = \det(V \cdot \mathrm{diag}(w_1^2, w_2^2, \ldots w_k^2) \cdot V^T) = \prod_{i=1}^{k} w_i^2$$

Thus, $GALI_k$ is computed by:

$$GALI_k = \sqrt{\det(A \cdot A^T)} = \prod_{i=1}^{k} w_i \Rightarrow \log\left(GALI_k\right) = \sum_{i=1}^{k} \log\left(w_i\right)$$

Behavior of $GALI_k$ for chaotic motion

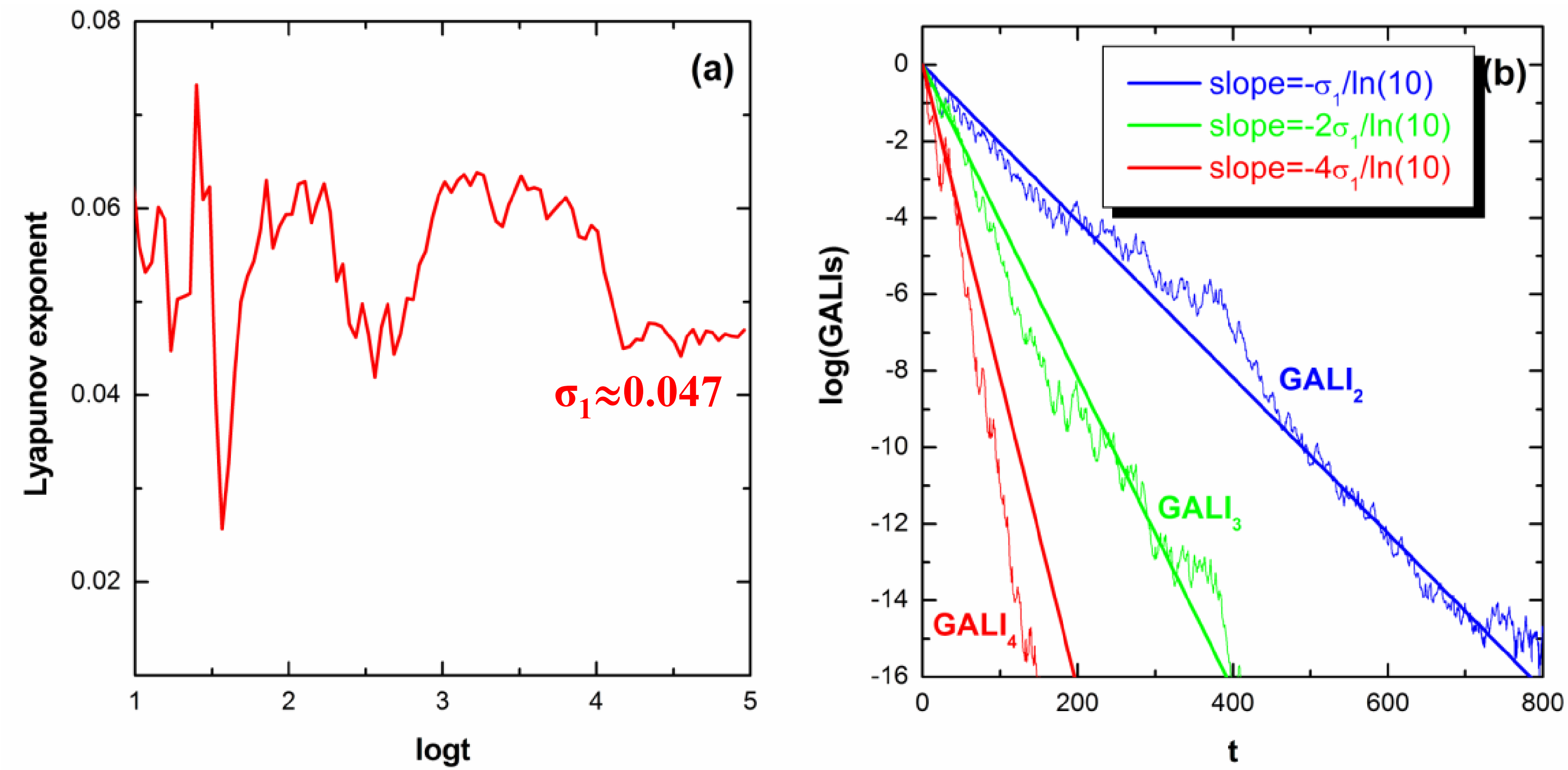
$GALI_k$ ($2 \le k \le 2N$) tends exponentially to zero with exponents that involve the values of the first k largest Lyapunov exponents $\sigma_1, \sigma_2, \ldots, \sigma_k$:

$$GALI_k(t) \propto e^{-\left[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_k)\right]t}$$

The above relation is valid even if some Lyapunov exponents are equal, or very close to each other.

Behavior of $GALI_k$ for chaotic motion

2D Hamiltonian (Hénon-Heiles system)

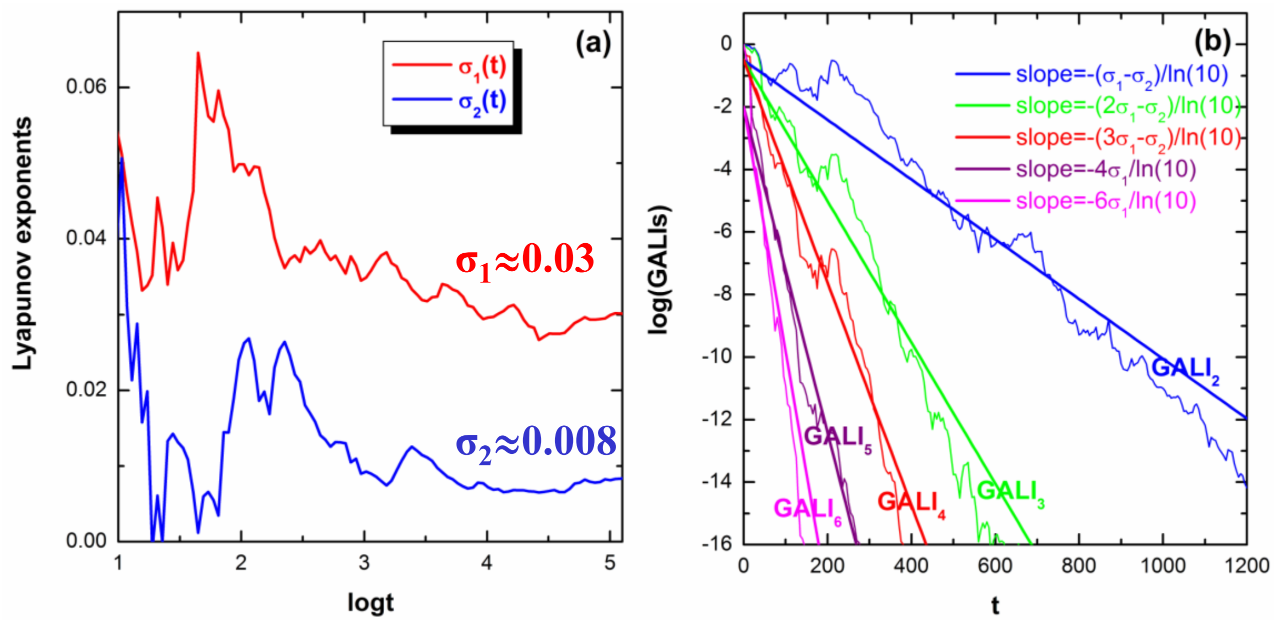


Behavior of $GALI_k$ for chaotic motion

3D system:

$$H_3 = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with ω_1=1, ω_2=$\sqrt{2}$, ω_3=$\sqrt{3}$, H_3=0.09.

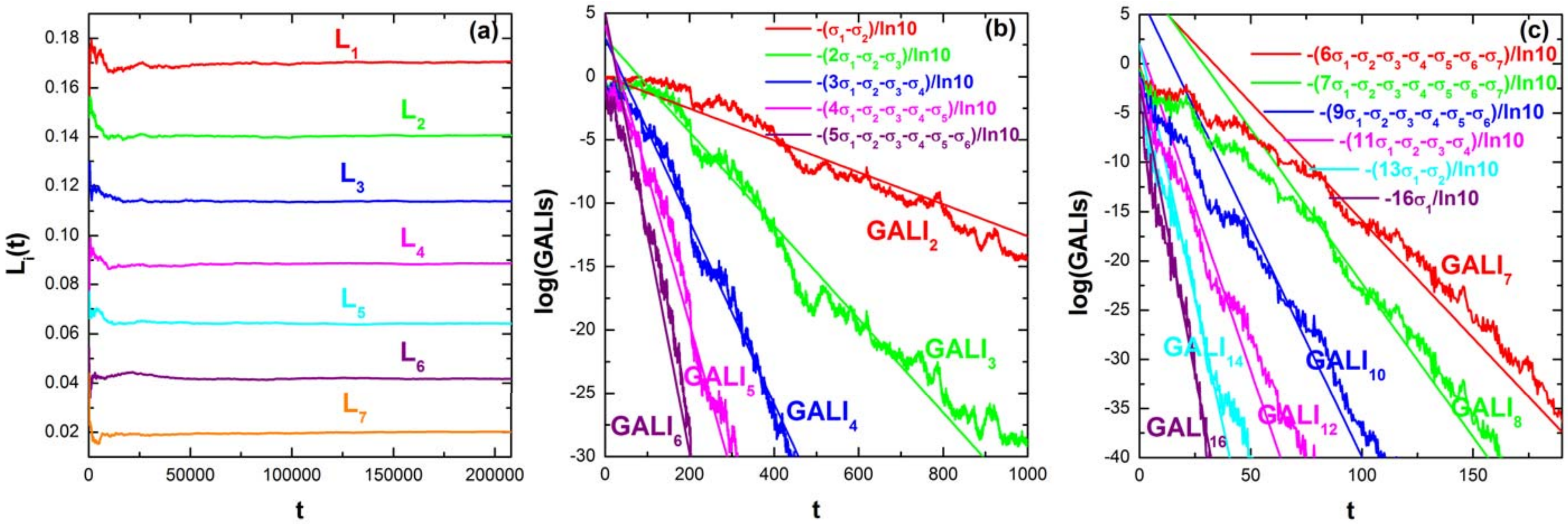


Behavior of $GALI_k$ for chaotic motion

N particles Fermi-Pasta-Ulam (FPU) system:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=0}^{N}\left[\frac{1}{2}(q_{i+1} - q_i)^2 + \frac{\beta}{4}(q_{i+1} - q_i)^4\right]$$

with fixed boundary conditions, N=8 and β=1.5.



Behavior of $GALI_k$ for regular motion

If the motion occurs on an s-dimensional torus with $s \leq N$ then the behavior of $GALI_k$ is given by (Skokos et al., 2008, EPJ-ST, 165, 5):

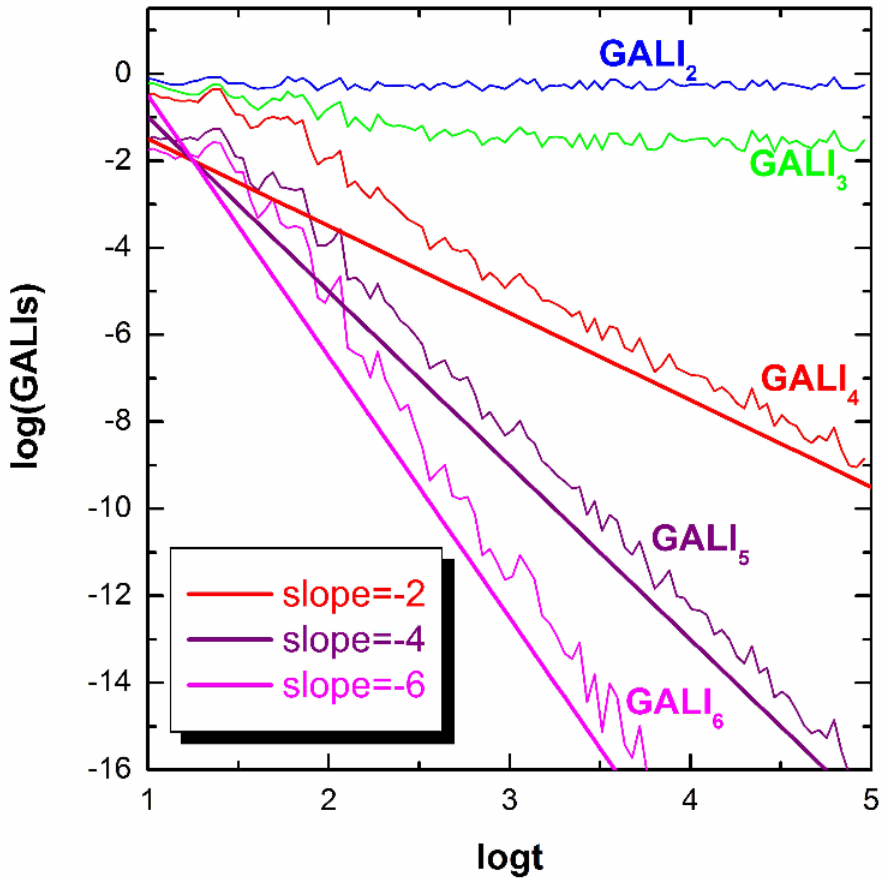
$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \leq k \leq s \\ \dfrac{1}{t^{k-s}} & \text{if} \quad s < k \leq 2N - s \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad 2N - s < k \leq 2N \end{cases}$$

while in the common case with s=N we have :

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \leq k \leq N \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad N < k \leq 2N \end{cases}$$

Behavior of $GALI_k$ for regular motion

3D Hamiltonian



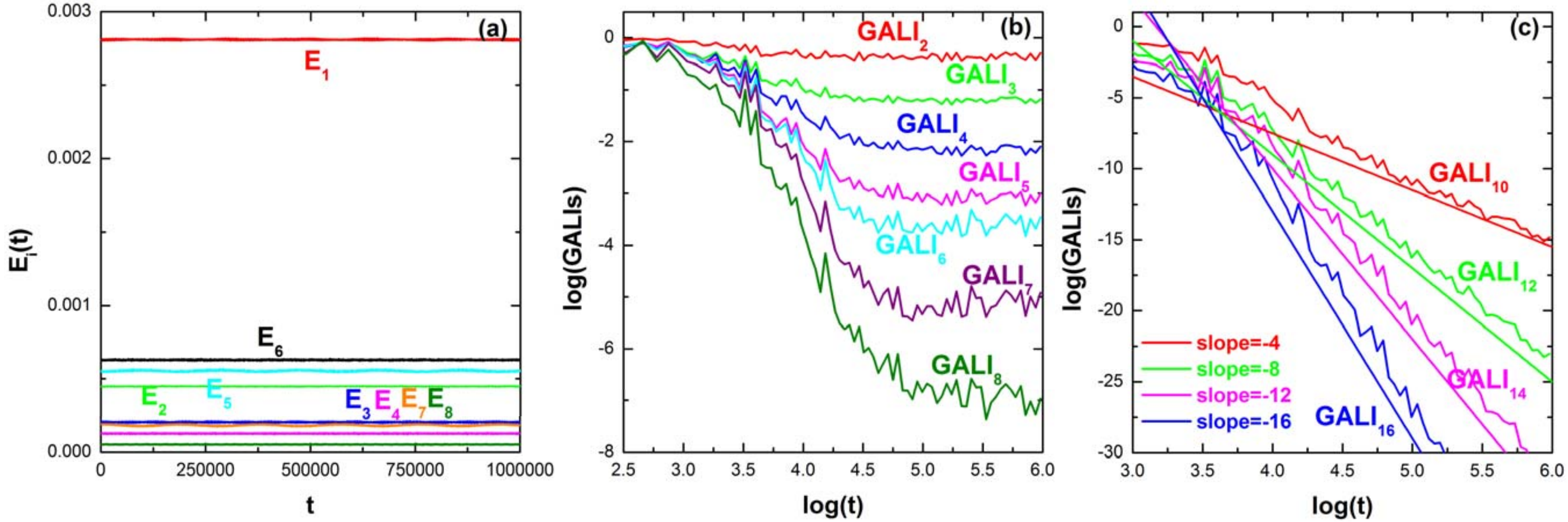
Behavior of GALI$_k$ for regular motion

N=8 FPU system: The unperturbed Hamiltonian (β=0) is written as a sum of the so-called harmonic energies E_i:

$$E_i = \frac{1}{2}\left(P_i^2 + \omega_i^2 Q_i^2\right),\ i = 1, \ldots, N$$

with:

$$Q_i = \sqrt{\frac{2}{N+1}}\sum_{i=1}^{N} q_i \sin\left(\frac{ki\pi}{N+1}\right),\ P_i = \sqrt{\frac{2}{N+1}}\sum_{i=1}^{N} p_i \sin\left(\frac{ki\pi}{N+1}\right),\ \omega_i = 2\sin\left(\frac{i\pi}{2(N+1)}\right)$$

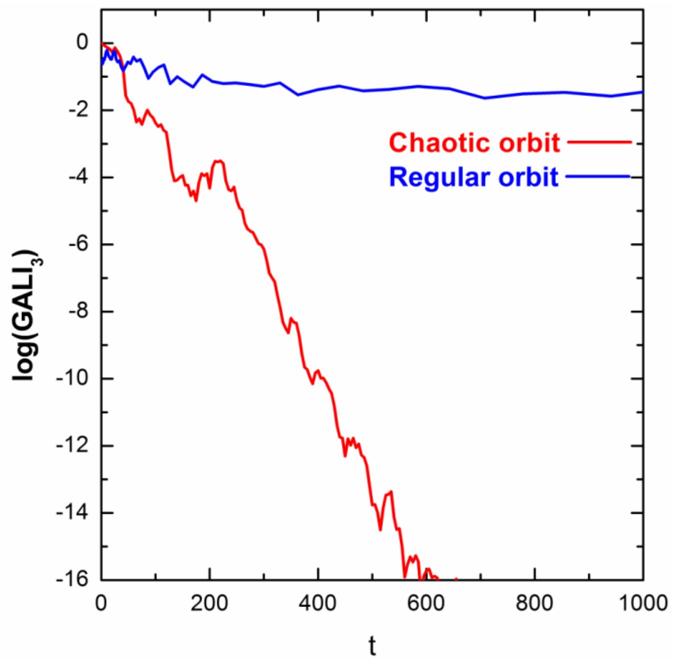


Global dynamics

- **GALI$_2$** (practically equivalent to the use of SALI)
- **GALI$_N$**

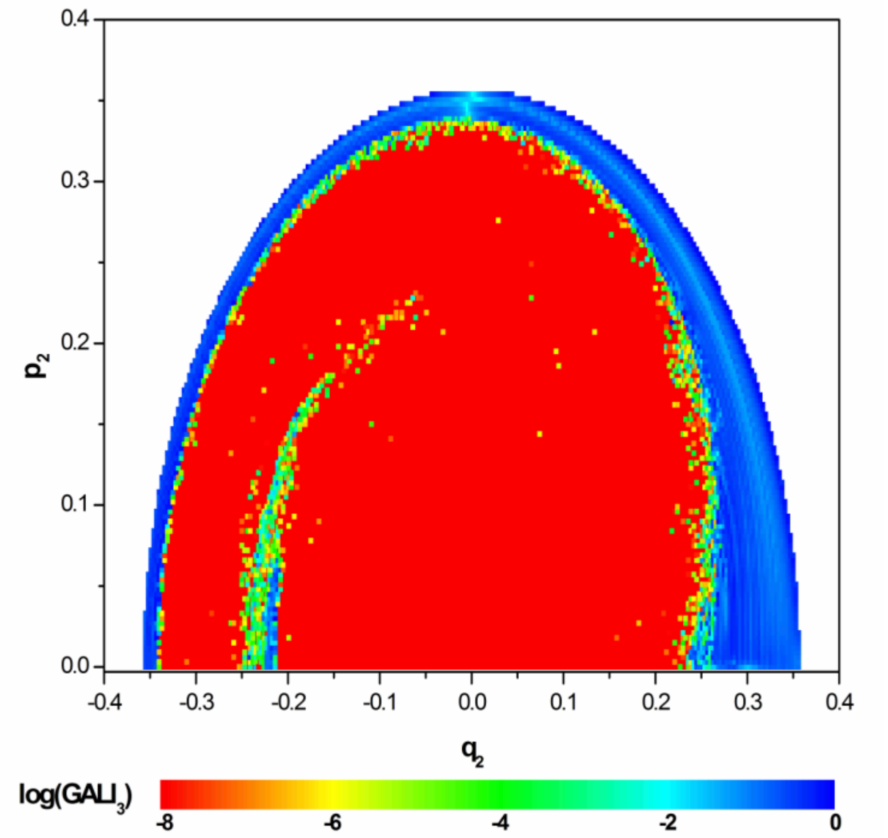
Chaotic motion: GALI$_N \rightarrow 0$ (exponential decay)

Regular motion: GALI$_N \rightarrow$ constant$\neq 0$



3D Hamiltonian

Subspace $q_3=p_3=0$, $p_2\geq 0$ for t=1000.



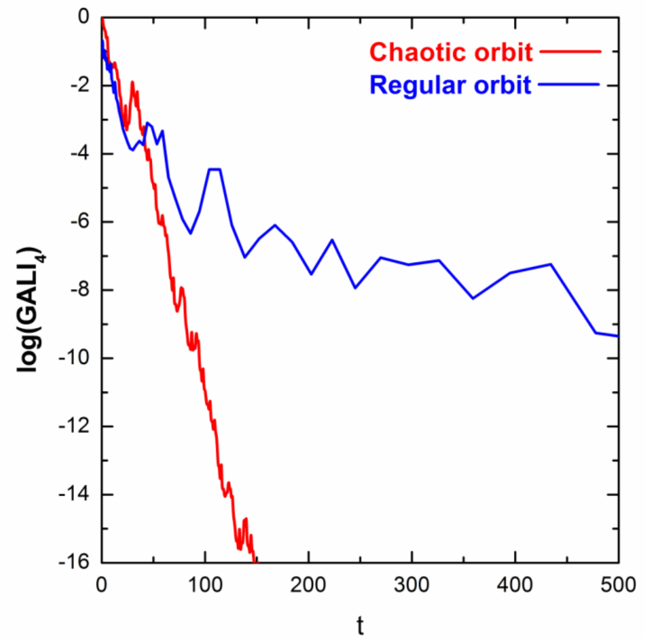
Global dynamics

GALI$_k$ with k>N

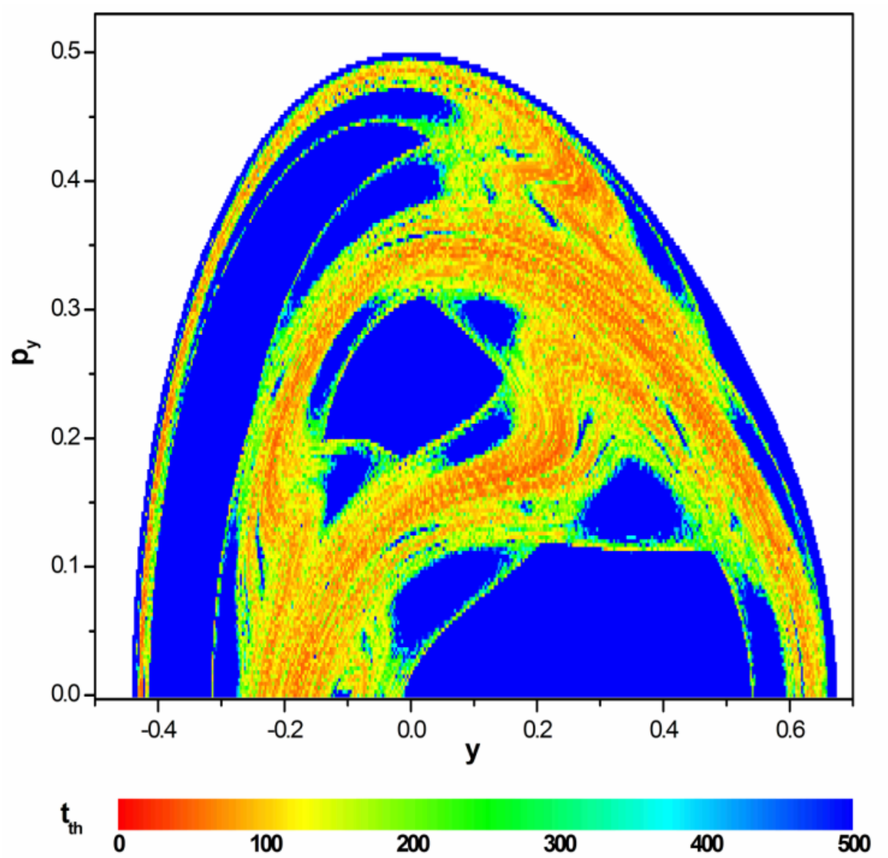
The index tends to zero both for regular and chaotic orbits but with completely different time rates:

Chaotic motion: exponential decay

Regular motion: power law

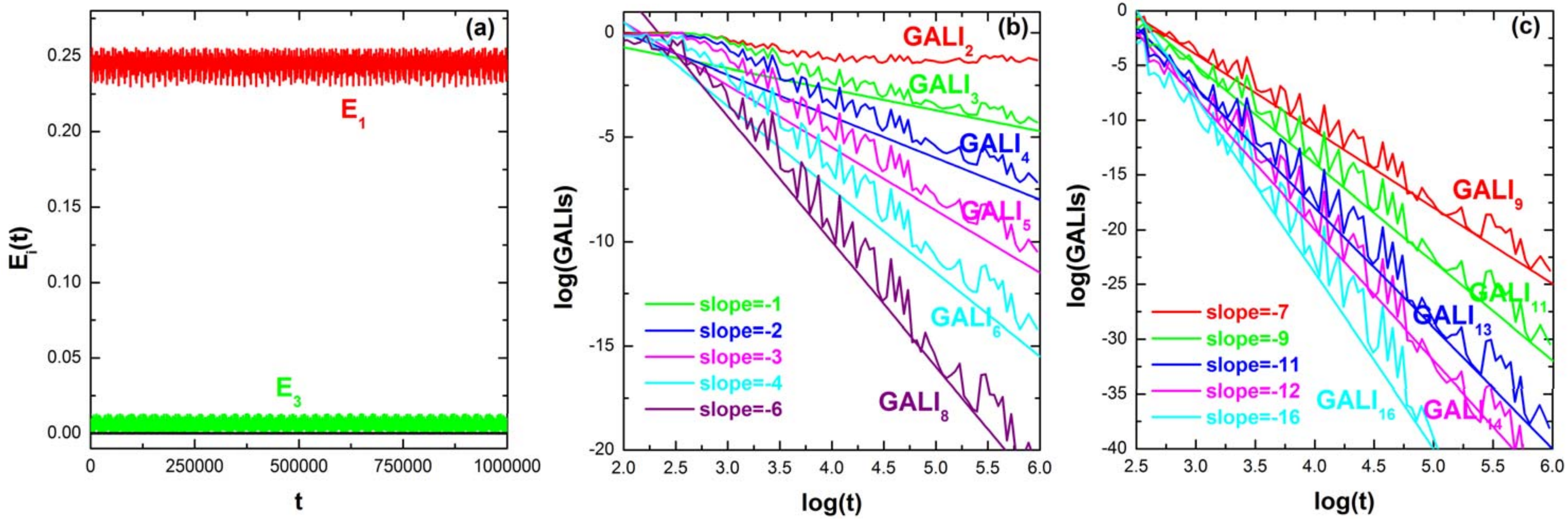


2D Hamiltonian (Hénon-Heiles)
Time needed for GALI$_4$<10^{-12}



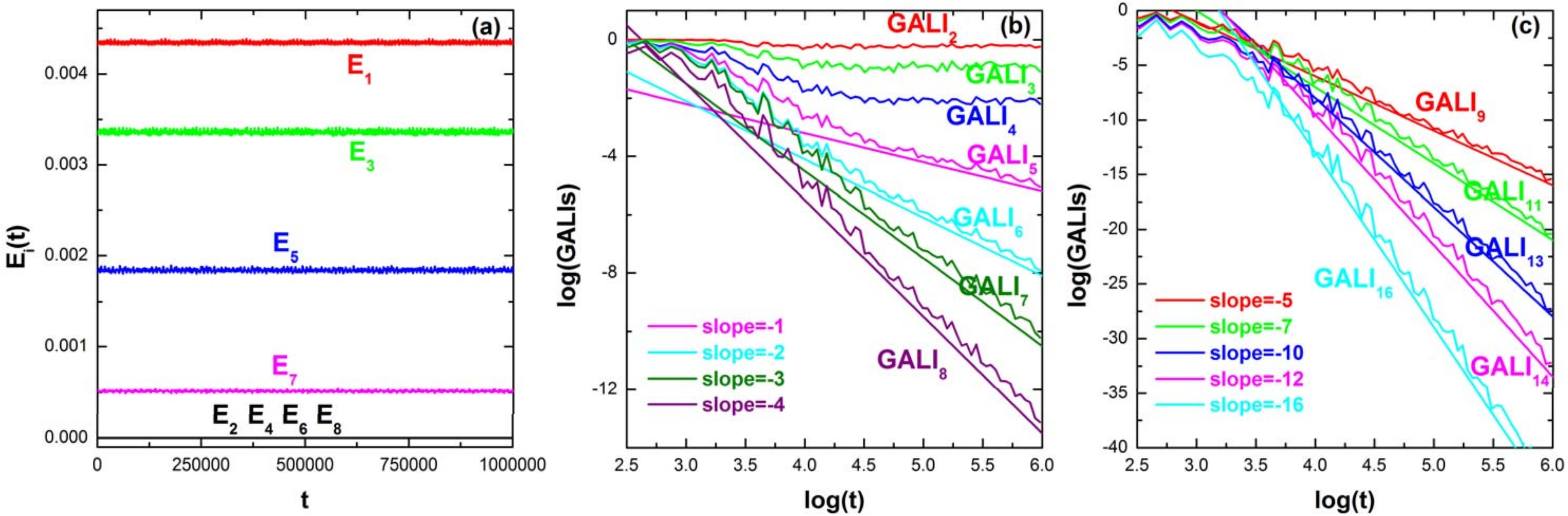
Regular motion on low-dimensional tori

A regular orbit lying on a 2-dimensional torus for the N=8 FPU system.



Regular motion on low-dimensional tori

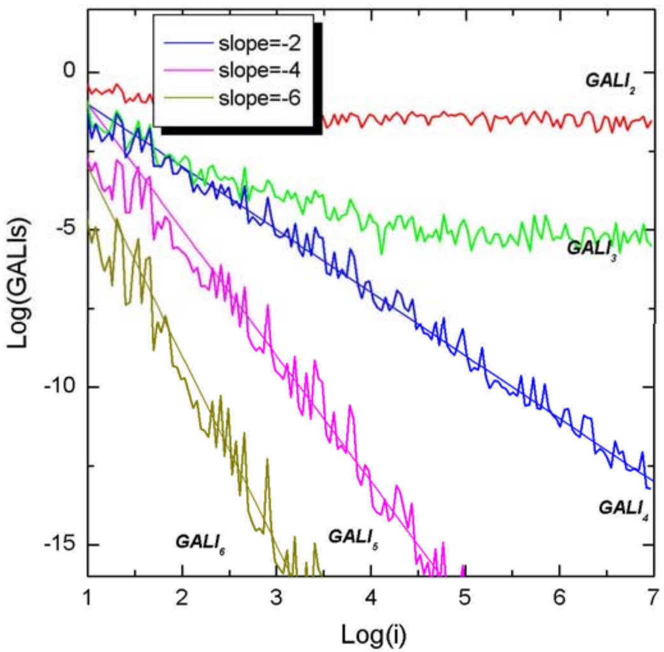
A regular orbit lying on a 4-dimensional torus for the N=8 FPU system.



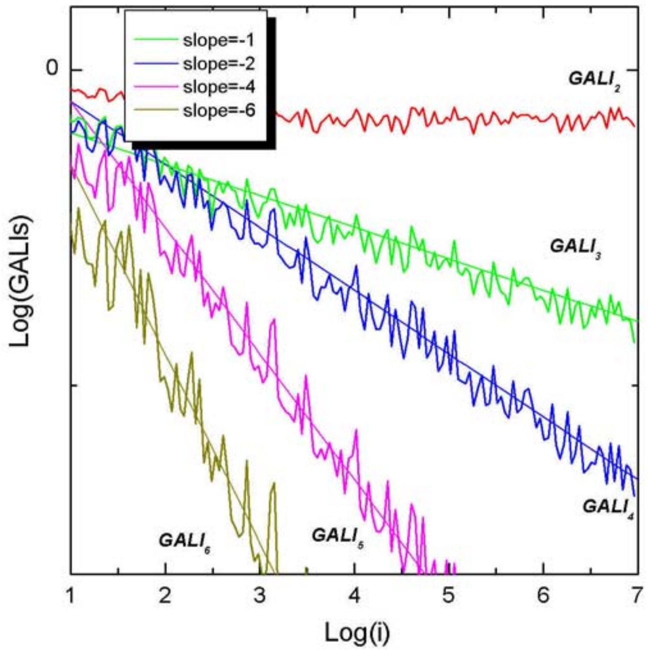
Low-dimensional tori - 6D map

$$\begin{aligned}
x_1' &= x_1 + x_2' \\
x_2' &= x_2 + \tfrac{K_1}{2\pi}\sin(2\pi x_1) - \tfrac{B}{2\pi}\{\sin[2\pi(x_5 - x_1)] + \sin[2\pi(x_3 - x_1)]\} \\
x_3' &= x_3 + x_4' \\
x_4' &= x_4 + \tfrac{K_2}{2\pi}\sin(2\pi x_3) - \tfrac{B}{2\pi}\{\sin[2\pi(x_1 - x_3)] + \sin[2\pi(x_5 - x_3)]\} \quad (\text{mod } 1) \\
x_5' &= x_5 + x_6' \\
x_6' &= x_6 + \tfrac{K_3}{2\pi}\sin(2\pi x_5) - \tfrac{B}{2\pi}\{\sin[2\pi(x_3 - x_5)] + \sin[2\pi(x_1 - x_5)]\}
\end{aligned}$$

3D torus



2D torus



Behavior of $GALI_k$

Chaotic motion:

$GALI_k \rightarrow 0$ exponential decay

$$GALI_k(t) \propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_k)]t}$$

Regular motion:

$GALI_k \rightarrow$ constant $\neq 0$ or $GALI_k \rightarrow 0$ power law decay

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \leq k \leq s \\ \dfrac{1}{t^{k-s}} & \text{if} \quad s < k \leq 2N-s \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad 2N-s < k \leq 2N \end{cases}$$

Conclusions

- Generalizing the SALI method we define the Generalized ALignment Index of order k ($GALI_k$) as the volume of the generalized parallelepiped, whose edges are k unit deviation vectors. $GALI_k$ is computed as the product of the singular values of a matrix (SVD algorithm).
- Behaviour of $GALI_k$:
  - ✓ Chaotic motion: it tends exponentially to zero with exponents that involve the values of several Lyapunov exponents.
  - ✓ Reguler motion: it fluctuates around non-zero values for $2 \leq k \leq s$ and goes to zero for $s < k \leq 2N$ following power-laws, with s being the dimensionality of the torus.
- $GALI_k$ indices :
  - ✓ can distinguish rapidly and with certainty between regular and chaotic motion
  - ✓ can be used to characterize individual orbits as well as "chart" chaotic and regular domains in phase space.
  - ✓ are perfectly suited for studying the global dynamics of multidimentonal systems
  - ✓ can identify regular motion in low–dimensional tori

References

- **SALI**
  - ✓ Skokos Ch. (2001) J. Phys. A, 34, 10029
  - ✓ Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2003) Prog. Theor. Phys. Supp., 150, 439
  - ✓ Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2004) J. Phys. A, 37, 6269
- **GALI**
  - ✓ Skokos Ch., Bountis T. C. & Antonopoulos Ch. (2007) Physica D, 231, 30-54
  - ✓ Skokos Ch., Bountis T. C. & Antonopoulos Ch. (2008) Eur. Phys. J. Sp. Top., 165, 5-14